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CORRELATION AFTER ASYMMETRICAL CLIPPING(U) ADMIRALTY  
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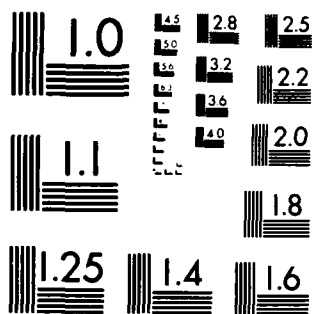
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## CORRELATION AFTER ASYMMETRICAL CLIPPING

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CORRELATION AFTER ASYMMETRICAL CLIPPING

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D E Weston

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Part of this paper is based on extracts from a 1975 paper

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CORRELATION AFTER ASYMMETRICAL CLIPPINGABSTRACT

1. A general formula is derived for the correlation coefficient between clipped waveforms or among detection sequences, for the case where the clipping is asymmetric or the detection probability departs from 50%. The analytic arcsine law for symmetrical clipping is rehearsed and new analytic forms are found for asymmetrical clipping with high positive correlation, numerically low correlation and high negative correlation.

### Introduction

2. This paper is about hard clipping, where a waveform or data stream is clipped about some level and converted into a binary sequence. The auto-correlation function of this sequence is considered, and in particular its relation to the correlation function of the original signal. For symmetrical clipping, ie clipping about the mean level, the well-known arcsine law was originally derived by Van Vleck in 1943 (Ref 1), and in acknowledging this it seems unnecessary to distinguish between auto-correlation and cross-correlation. Other sources are Refs 2 and 3. Here we extend the calculation to clipping at arbitrary levels, where it is believed the result has not been published before. The calculation is quite straightforward but the answer has some unexpected features, and is also one that is needed.

3. There could be some interest in asymmetrical clipping in signal processing itself. But the real interest arises from the important analogy between target detection theory and data processing on the one hand, and clipped signal processing on the other hand. Thus sonar and radar detection decisions are commonly or effectively based on whether or not a threshold is crossed. The precise proportion of signals crossing the threshold cannot be controlled, so we cannot easily arrange a 50% probability  $P$  corresponding to the arcsine law. In practice detection is often a cumulative process with a low probability  $P$  for any given signal and a wide departure from the arcsine law. The present paper stands on its own but also serves as a companion or back-up to Ref 4, which uses the asymmetrical clipping relations.



4.

4. Here we first set the scene by recapitulating in section I on some descriptors for the scatter diagram, as a lead in to a particularly simple derivation of the arcsine law in section II. Section III is the main one, presenting the general correlation answers. Sections IV - VI derive analytic forms which apply in various limiting conditions.

#### I CORRELATION AND THE SCATTER DIAGRAM

5. Let  $x$  and  $y$  be random variates of zero mean, normalised if necessary in order to have the same standard deviation. Figure 1 represents the scatter diagram or joint probability density function. The line  $y = x$  is shown as the major axis, with half axis length

$$\sqrt{(y+x)^2} = \sqrt{\bar{x}^2 + \bar{y}^2 + 2\bar{x}\bar{y}} = \sqrt{2\bar{x}^2(1+\rho)}. \quad \dots\dots (1)$$

The line  $y = -x$  or minor axis has half length

$$\sqrt{(y-x)^2} = \sqrt{\bar{x}^2 + \bar{y}^2 - 2\bar{x}\bar{y}} = \sqrt{2\bar{x}^2(1-\rho)}. \quad \dots\dots (2)$$

Thus the ratio of the axes can be written in terms of correlation coefficient  $\rho$ ,

$$E = \sqrt{\frac{1-\rho}{1+\rho}}, \quad \dots\dots (3)$$

equivalent to

$$\rho = \frac{1 - E^2}{1 + E^2}. \quad \dots\dots (4)$$

The asymptotic form for  $|\rho|$  small, holding up quite well as  $|\rho|$  increases, is

$$\rho + E \sim 1. \quad \dots\dots (5)$$

Using these relations an eye fit of an ellipse to a scatter diagram can rapidly give an approximate value for the correlation coefficient.

## II SYMMETRICAL CLIPPING AND THE ARCSINE LAW

6. If the detection decisions or clipper outputs  $x_1$  and  $y_1$  are represented as either 0 or 1, symmetrical clipping will be about a mean level 0.5. In other words probability of threshold crossing  $P$  is 50%.

Let us shift the means so that

$$\bar{x}_1 = \bar{y}_1 = 0, \quad \text{..... (6)}$$

$$\overline{x_1^2} = \overline{y_1^2} = 0.25 \quad \text{..... (7)}$$

Taking these numerical values is useful for the sake of being specific, but in fact they cancel out in equation (8) below. There will be a positive correlator output ( $x_1 y_1 = 0.25$ ) for the probability masses AA in two quadrants of figure 1, with a negative output ( $x_1 y_1 = -0.25$ ) for BB. The output correlation coefficient after clipping

$$r = \frac{\overline{x_1 y_1}}{\sqrt{\overline{x_1^2} \overline{y_1^2}}} = \frac{0.25(2A - 2B)}{0.25} = 1 - 4B. \quad \text{..... (8)}$$

7. To calculate  $B$  it is usual to make some assumptions about the distribution of  $x$  and  $y$ , in fact to assume they are both normal, and then integrate. Here we will assume instead that the probability contours are all elliptical, as discussed further below. The trick now is to distort the scatter diagram by expanding it in the ( $y = -x$ ) direction by the factor  $E^{-1}$ , so that the ellipses become circles, figure 2. This does not change the relative magnitude of  $A$  and  $B$ . The original  $\pi/2$  quadrant angle for  $B$  now becomes  $2\alpha$  where

$$\tan \alpha = E = \sqrt{\frac{1-\rho}{1+\rho}}, \quad \text{..... (9)}$$

$$\text{giving } \alpha = \arcsin \sqrt{\frac{1-\rho}{2}} \quad \text{..... (10)}$$

6.

8. Since we have central symmetry the probability associated with a sector is directly proportional to sector angle, and B is merely  $\alpha/\pi$ .

Equation (8) becomes

$$r = 1 - \frac{4}{\pi} \arcsin \sqrt{\frac{1-\rho}{2}}, \quad \dots (11)$$

or 
$$\left(\frac{1-r}{2}\right) = \frac{2}{\pi} \arcsin \sqrt{\frac{1-\rho}{2}}, \quad \dots (12)$$

or 
$$r = \frac{2}{\pi} \arcsin \rho. \quad \dots (13)$$

9. Equation (13) is the usual form of the so-called arcsine law.

Together with the form (12) it can be used to derive the limiting forms -

$$\rho, r \ll 1 : \quad r \sim (2/\pi)\rho. \quad \dots (14)$$

$$(1-\rho), (1-r) \ll 1 : \quad (1-r) \sim (2/\pi) \sqrt{2(1-\rho)}. \quad \dots (15)$$

10. The basis of this derivation is that the original contours are elliptical. It also seems that any other contour shape must lead to a law different from (13). Now the elliptical contours come from the product of the laws of probability distribution with distance  $d$  along the  $(y = x)$  and  $(y = -x)$  directions, whereas the equation for an ellipse involves the weighted sums of the  $d^2$  values. These points can only be reconciled if the distribution laws have an exponential with  $d^2$  terms, ie the normal or Gaussian law. The sums  $(y+x)$  and  $(y-x)$  therefore need to be Gaussian, and since suitable combinations of the  $(y+x)$  and  $(y-x)$  laws give the  $x$  and  $y$  laws these too need to be normal. Thus our derivation shows that the normality assumption is implicit in the arcsine law (13).

### III ASYMMETRICAL CLIPPING AND THE GENERAL QUADRANT FORMULA

11. In the general case clipping is about an arbitrary level. Figure 3 illustrates this for positive correlation and for clipping such that  $P > 0.5$ , at the same level for  $x$  and  $y$ . We keep the mean values  $\bar{x}_1$  and  $\bar{y}_1$  zero as in equation (6) and find this leads to

$$\bar{x}_1^2 = \bar{y}_1^2 = (1-P)P^2 + P(1-P)^2 = P(1-P) . \quad \dots (16)$$

12. We need to know the probability masses in the quadrants B, C and D defined with respect to the clipping levels. Normal probability is assumed. If  $L(x, y, \rho)$  refers to the integrated joint probability in the quadrant where the  $x$  and  $y$  values exceed the stated values we can use the rules in Ref 5 to write

$$\begin{aligned} B &= L(x, -x, -\rho) \\ &= L\left(x, 0, \sqrt{\frac{1-\rho}{2}}\right) - L\left(x, 0, -\sqrt{\frac{1-\rho}{2}}\right) \\ &= L(+) - L(-) . \end{aligned} \quad \dots (17)$$

$$C = L(x, x, \rho) = 2L(-) . \quad \dots (18)$$

$$D = L(-x, -x, \rho) = 1 - 2L(+) . \quad \dots (19)$$

13. The multiplication term or covariance after clipping

$$\begin{aligned} \overline{x_1 y_1} &= CP^2 + D(1-P)^2 - 2BP(1-P) \\ &= 2PL(-) - 2(1-P)L(+) + (1-P)^2 . \end{aligned} \quad \dots (20)$$

14. The correlation after clipping becomes

$$r = \frac{\overline{x_1 y_1}}{\bar{x}_1^2} = \frac{2L(-)}{1-P} - \frac{2L(+)}{P} + \frac{1-P}{P} . \quad \dots (21)$$

8.

15. Sample curves are plotted in figure 4 using L values from Ref 5. They are labelled with the positive x levels for the clipping, but symmetry means that we get the same curves for negative values of x. The P values or percentage clippings then become 50, 31, 16 and 7 - perhaps of more practical interest.

16. The calculations are for identical x and y clipping levels, and this is the common case, encountered for example in auto-correlation. Calculations with different levels should be similar and straightforward. As a special case note that if the x and y levels are equal in magnitude but opposite in sign we merely have to interchange the plots in the two quadrants in figure 4.

17. The behaviour in figure 4 is surprisingly complicated, and some of the limiting forms are discussed in the next three sections. We confirm now that for clipping at  $x=0$  and  $P=0.5$  we do revert to the arcsine law of equation (13).

18. The wave spectrum is of course the Fourier transform of the correlation function, and we remark here on the ordinary spectrum and the auto-correlation function. It may be remembered that symmetrical clipping produces a relative shift towards the higher spectral frequencies. We point out that with asymmetrical clipping this shift is reinforced, as a consequence of the behaviour for both positive and negative correlation.

#### IV ASYMMETRICAL CLIPPING WITH HIGH POSITIVE CORRELATION

18. Figure 3 also serves to illustrate the limiting case where the correlation is very high. The scatter ellipse then becomes very narrow, and the probability B falls off from its value in the symmetrical case as the probability density function Z [ cf Ref 3 p 493 ],

$$B = Z \arcsin \sqrt{\frac{1-\rho}{\pi}} . \quad \dots (22)$$

The probability masses C and D will depend on the percentage detection, ie the probability integral P, with only a small correction for B -

$$C = (1-P) - B , \quad \dots (23)$$

$$D = P - B . \quad \dots (24)$$

20. The multiplication term (compare equation (20))

$$\overline{x_1 y_1} = P(1-P) - B . \quad \dots (25)$$

21. Taking the normalisation from equation (16) the coefficient after clipping becomes

$$r = \frac{\overline{x_1 y_1}}{\overline{x_1^2}} = 1 - \frac{Z}{P(1-P)} \arcsin \sqrt{\frac{1-\rho}{\pi}} . \quad \dots (26)$$

Remembering that  $(1-\rho)$  and  $(1-r)$  are small gives

$$(1-r) = \frac{Z}{P(1-P)} \sqrt{\frac{1-\rho}{\pi}} . \quad \dots (27)$$

This formula is consistent with the general result (21) and with the limiting form (15) for symmetrical clipping.

#### V ASYMMETRICAL CLIPPING WITH NUMERICALLY LOW CORRELATION

22. For low correlation the scatter diagram elliptical contours become nearly circular. To deal with this case we replace the scatter diagram by an allied form having very narrow elliptical contours, figure 5.

We define three new independent variables abc such that

$$\overline{x^2} = \overline{y^2} = \overline{a^2} = \overline{b^2} = \overline{c^2} . \quad \dots (28)$$

10.

Choose  $x$  and  $y$  so that

$$x = \sqrt{\rho} a + \sqrt{1-\rho} b, \quad \dots (29)$$

$$y = \sqrt{\rho} a + \sqrt{1-\rho} c. \quad \dots (30)$$

Here  $\rho$  is introduced as an arbitrary constant but is found to be equal to the correlation coefficient between  $x$  and  $y$ . For convenience it is treated as positive in the discussion, but the formulae hold whatever its sign. In figure 5 the diagonal lines correspond to fixed values of  $x$ , and therefore to possible clipping points.

23. Take the general case with asymmetrical clipping, the clipping level being defined by the overall probability  $P$ . Let  $\sqrt{\rho} a$  have the small fixed positive value  $\beta$ , this will increase the chance  $\mu$  of exceeding the clipping level even though the  $\sqrt{1-\rho} b$  component is still allowed to vary. Since  $\beta$  is small the distribution function  $Z$  for the probability density may be taken as the value corresponding to  $P$ , and

$$\mu = P + \beta Z. \quad \dots (31)$$

(The lack of a linear  $\beta Z$  term in equation (32) below means that terms in  $\beta^2$  need not be included in equation (31)). The probability that both  $x$  and  $y$  will exceed the clipping level (and give  $x_1 y_1 = (1-P)^2$ ) is therefore  $\mu^2$ , the probability that neither will do so is  $(1-\mu)^2$  and that they will have opposite signs is  $2\mu(1-\mu)$ . Thus for our fixed  $\beta$

$$\begin{aligned} \overline{x_1 y_1} &= \mu^2 (1-P)^2 + (1-\mu)^2 P^2 - 2\mu(1-\mu)P(1-P) \\ &= \beta^2 Z^2. \quad \dots (32) \end{aligned}$$

This simple result brings out the square law dependence on  $\beta$ , any non-zero correlated component very reasonably leads to a positive correlation. We now take the average for all values of  $\beta$ ,

$$\overline{x_1 y_1} = \rho \overline{a^2} Z^2 . \quad \text{..... (33)}$$

24. Using equation (16) the clipped correlation coefficient

$$r = \frac{\overline{x_1 y_1}}{\overline{x_1^2}} = \frac{\rho \overline{a^2} Z^2}{P(1-P)} . \quad \text{..... (34)}$$

The Z and P effects are similar to those in equation (27), but not identical. If we now assume that the distribution is normal the ratio  $r/$  may be calculated as a function of P using standard tables, remembering that  $\overline{a^2}$  is the mean variance. Figure 6 shows the symmetry in the relation. The result is consistent with the general formula (21), and confirms that all the figure 4 lines do indeed go through the origin. For symmetrical clipping  $P = 0.5$  and  $Z = (2\pi \overline{a^2})^{-1/2}$ , so that the ratio becomes  $(2/\pi)$  in agreement with equation (14).

#### VI ASYMMETRICAL CLIPPING WITH HIGH NEGATIVE CORRELATION

25. High negative correlation is illustrated for  $P > 0.5$  in figure 7, which allows the immediate identification of the probability masses as

$$B = 1-P , \quad \text{..... (35)}$$

$$C = 0 , \quad \text{..... (36)}$$

$$D = 2P-1 . \quad \text{..... (37)}$$

26. We find for the limiting forms

$$\begin{aligned} \overline{x_1 y_1} &= - (1-P)^2 , \\ r &= - (1-P)/P . \end{aligned} \quad \text{..... (39)}$$

If  $P < 0.5$  the roles of C and D are interchanged, as are P and  $(1-P)$  in all these equations. Note that r is independent of  $\rho$  in all these limiting conditions. Equation (39) is consistent with the general formula (21), and helps in understanding the wide spread of the figure 4 lines.



12.

## VII CONCLUSIONS

27. The chief result is the general equation (21), illustrated by the plot in figure 4, with surprisingly complicated behaviour. There is consistency with the special or limiting forms for symmetrical clipping (13), high positive correlation (27), numerically low correlation (34) and high negative correlation (39).

### Acknowledgements

28. I have had the benefit of useful discussions with Dr A R Pratt and with many ARE colleagues.

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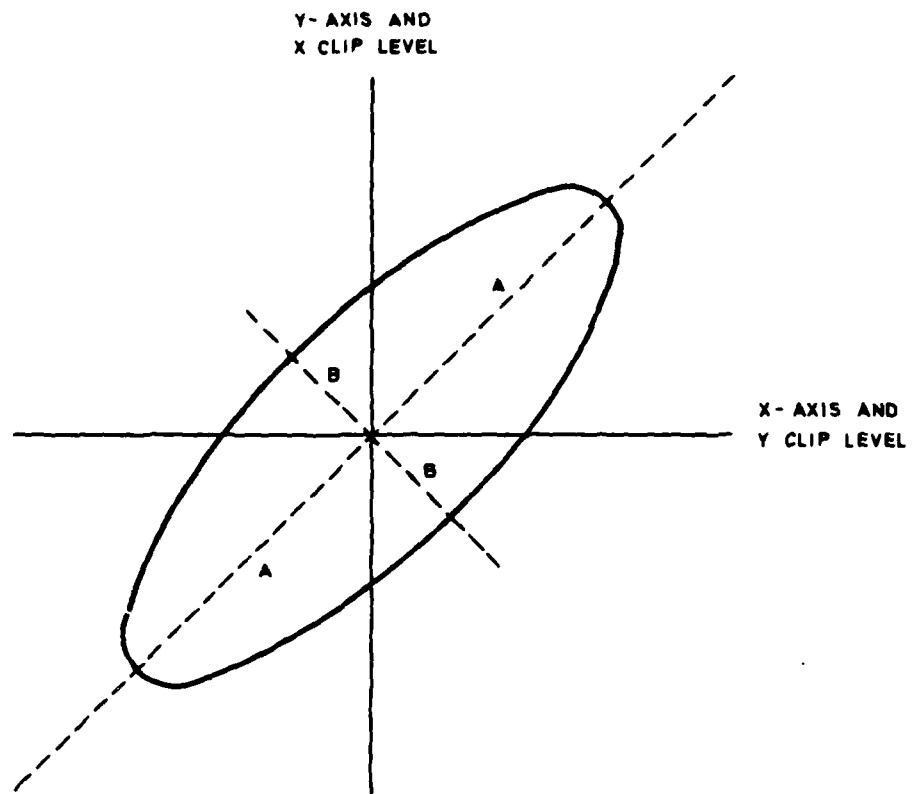


FIG. 1 SCATTER DIAGRAM WITH SYMMETRICAL CLIPPING

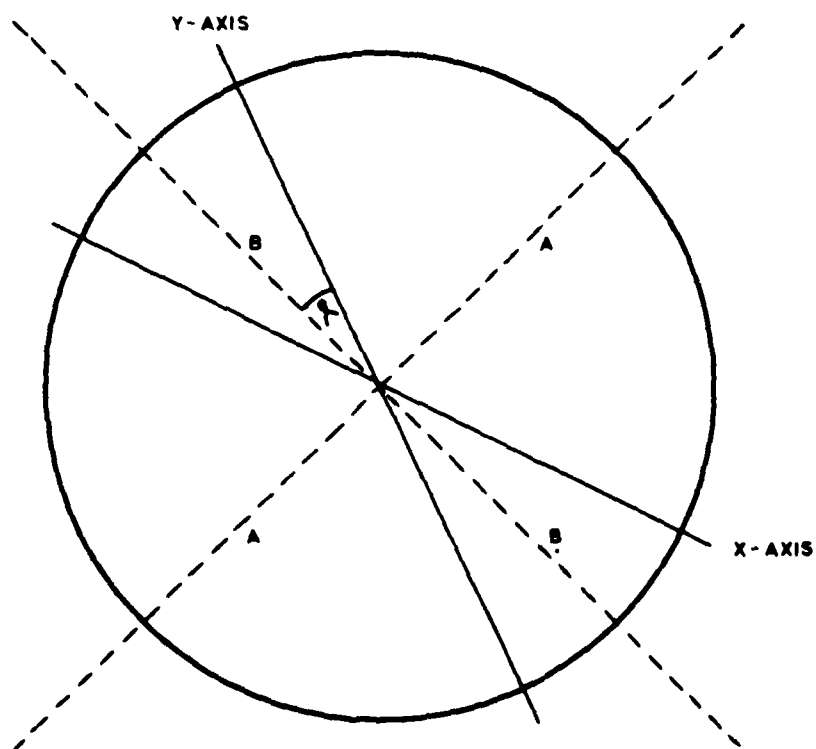


FIG. 2 DISTORTED SCATTER DIAGRAM WITH SYMMETRICAL CLIPPING

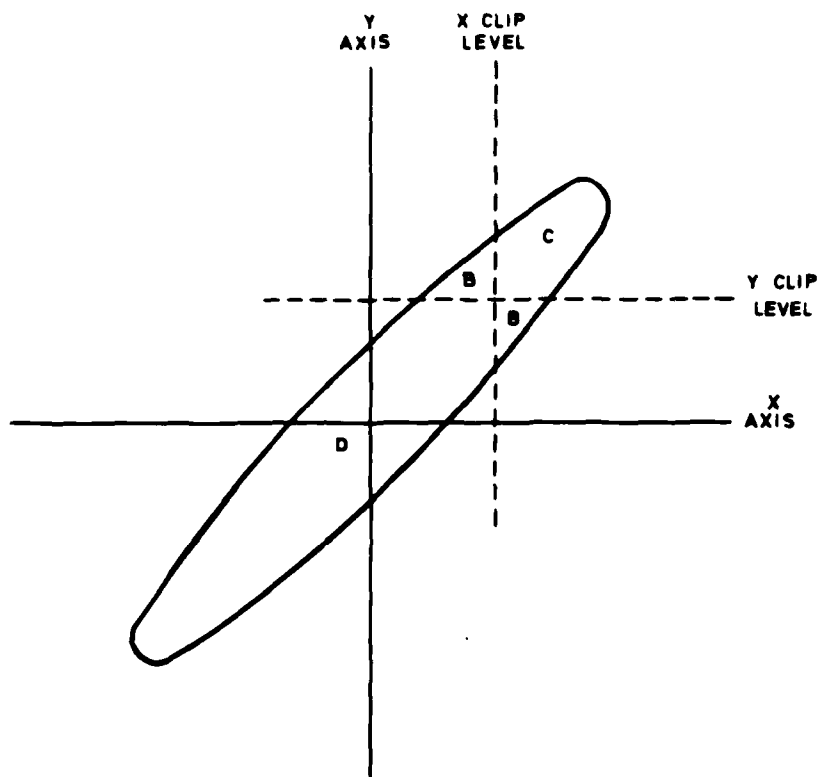


FIG 3 SCATTER DIAGRAM WITH  
ASYMMETRICAL CLIPPING, SHOWN FOR  $\rho > 0.5$

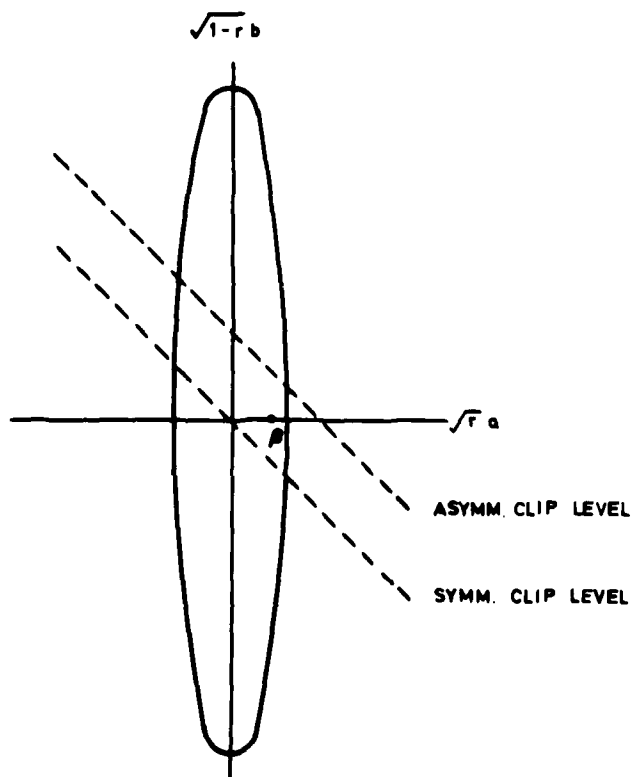


FIG 5 SCATTER DIAGRAM FOR CORRELATED AND UNCORRELATED  
COMPONENTS OF  $X$ , WITH CORRELATION LOW

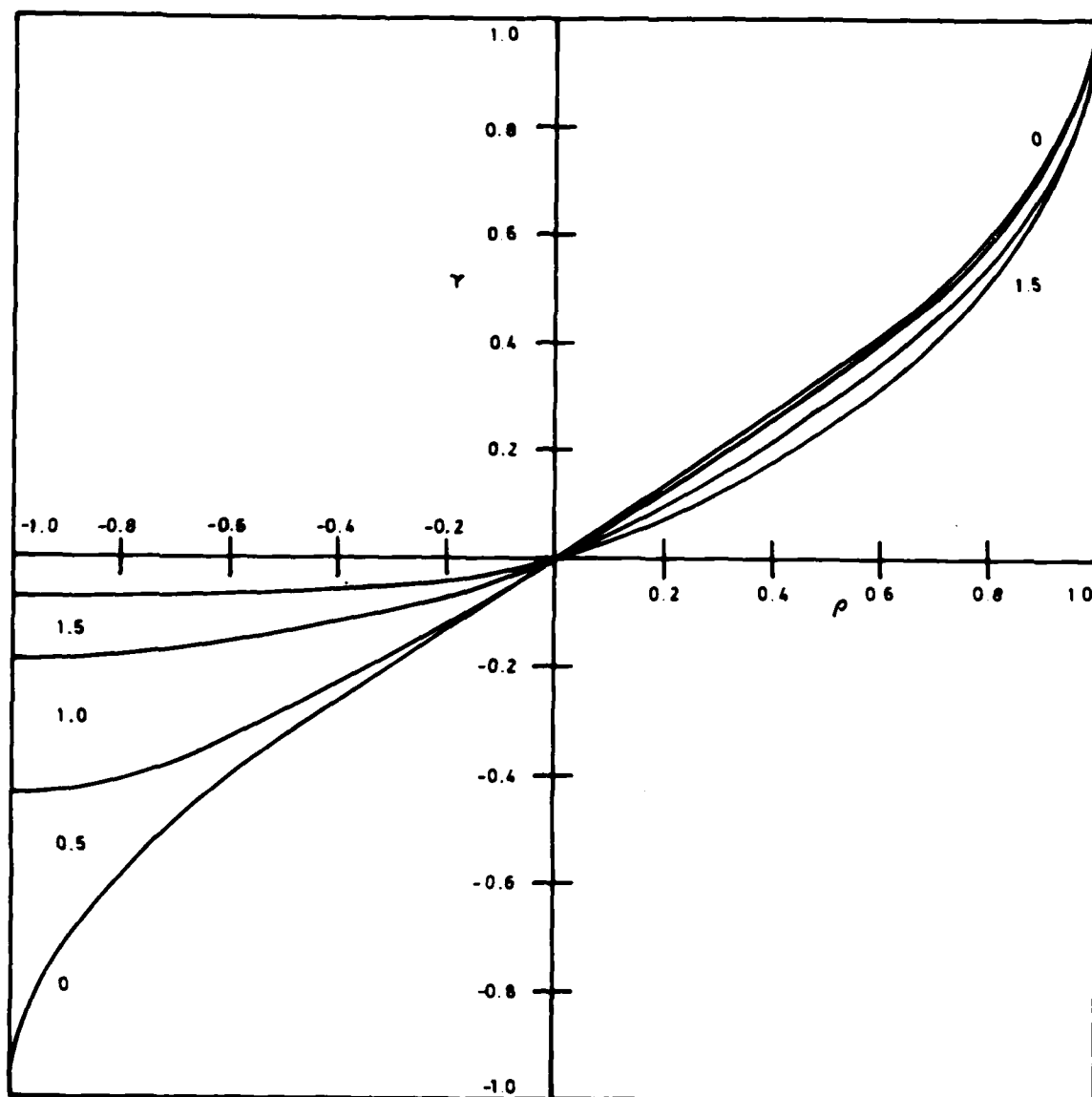


FIG 4 RELATION BETWEEN CLIPPED AND UNCLIPPED CORRELATION COEFFICIENTS  $\gamma$  AND  $\rho$  FOR ARBITRARY CLIPPING LEVEL THE MARKED  $X$  VALUES 0, 0.5, 1.0 AND 1.5 CORRESPOND TO 50, 69, 84 AND 93% CLIPPING

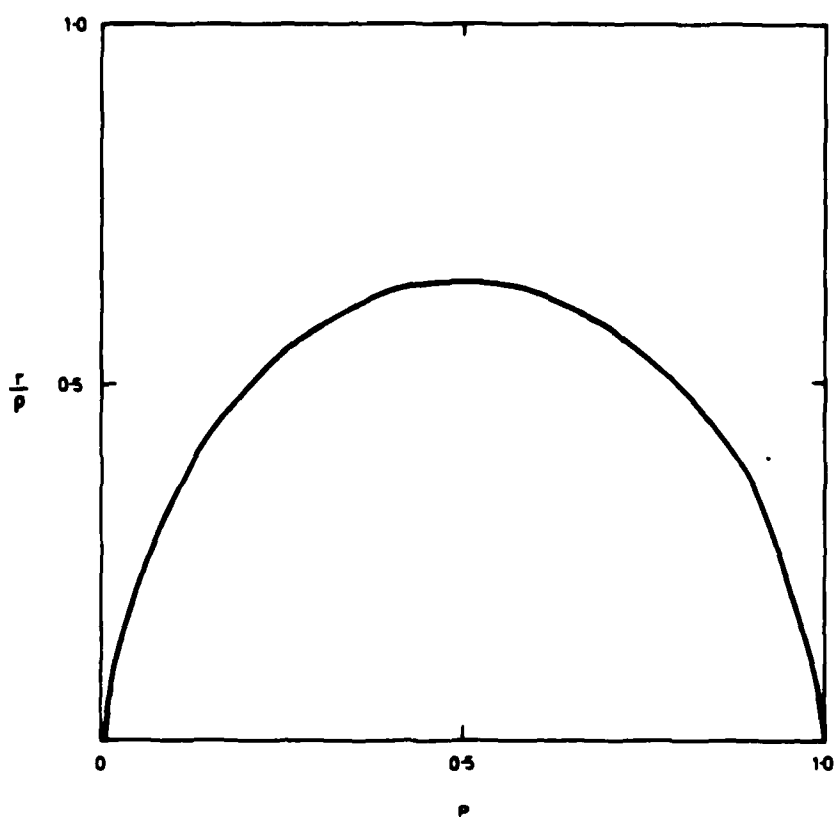


FIG. 6 RATIO  $\frac{r}{p}$  OF CORRELATION COEFFICIENTS (ASSUMED LOW) AFTER AND BEFORE CLIPPING (OR DETECTION) AS A FUNCTION OF CLIPPING LEVEL,  $p$  BEING PROBABILITY OF EXCEEDING CLIPPING LEVEL (OR OF DETECTION)

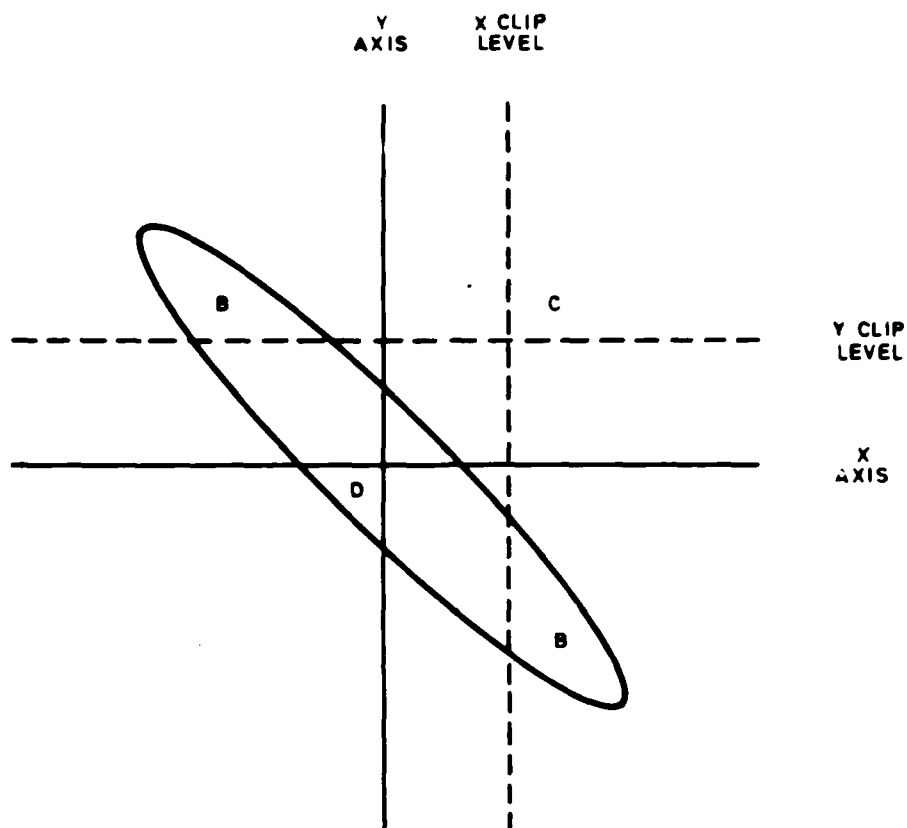


FIG 7 SCATTER DIAGRAM WITH NUMERICALLY HIGH NEGATIVE CORRELATION AND ASYMMETRICAL CLIPPING

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17. Abstract  A general formula is derived for the correlation coefficient between clipped waveforms or among detection sequences, for the case where the clipping is asymmetric or the detection probability departs from 50%. The analytic arcsine law for symmetrical clipping is rehearsed and new analytic forms are found for asymmetrical clipping with high positive correlation, numerically low correlation and high negative correlation.			



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